5-5 التكامل بالتجزيء

حاول أن تحل (1) : أوجد :

$$\int x\cos x\, dx$$

الحل :

$u = x \qquad dv = \cos x\, dx$

$du = dx \qquad v = \sin x$

$$\int u\,dv = uv - \int v\,du$$

$$\int x\cos x\, dx = x\sin x - \int \sin x\, dx$$

$$= x\sin x + \cos x + C$$

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حاول أن تحل (2) أوجد :

a) $$\int (x-3)e^{x-3}\, dx$$

الحل :

$u = x - 3 \qquad dv = e^{x-3}dx$

$du = dx \qquad v = e^{x-3}$

$$\int u\,dv = uv - \int v\,du$$

$$\int (x-3)e^{x-3}\, dx = (x-3)e^{x-3} - \int e^{x-3}dx$$

$$= (x-3)e^{x-3} - e^{x-3} + C = (x-4)e^{x-3} + C$$

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b) $$\int 4xe^{-5x}\, dx$$

الحل :

$u = 4x \qquad dv = e^{-5x}dx$

$du = 4dx \qquad v = -\frac{1}{5}e^{-5x}$

$$\int u\,dv = uv - \int v\,du$$

$$\int 4xe^{-5x}\, dx = 4x\left(-\frac{1}{5}e^{-5x}\right) - \int -\frac{4}{5}e^{-5x}dx$$

$$= -\frac{4}{5}x\, e^{-5x} - \frac{4}{25}e^{-5x} + C$$

حاول أن تحل(3) أوجد :

$$\int \ln x \ dx$$

الحل :

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}dx \qquad v = x$$

$$\int udv = uv - \int vdu$$

$$\int \ln x \ dx = x \ln x - \int x.\frac{1}{x}dx$$

$$= x \ln x - \int dx = x \ln(x) - x + C$$

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حاول أن تحل(4) أوجد :

$$\int (x+1)\ln(x+1) \ dx$$

الحل :

$$u = \ln(x+1) \qquad dv = (x+1)dx$$

$$du = \frac{1}{x+1}dx \qquad v = \frac{1}{2}x^2 + x$$

$$\int udv = uv - \int vdu$$

$$\int (x+1)\ln(x+1) \ dx = \left(\frac{1}{2}x^2 + x\right) \ln(x+1) - \int \frac{x^2+2x}{2}.\frac{1}{x+1}dx$$

$$= \left(\frac{1}{2}x^2 + x\right) \ln(x+1) - \int \frac{x^2+2x}{2x+2}dx$$

$$= \left(\frac{1}{2}x^2 + x\right) \ln(x+1) - \int \left(\frac{1}{2}x + \frac{1}{2} - \frac{1}{2x+2}\right)dx$$

$$= \left(\frac{1}{2}x^2 + x\right) \ln(x+1) - \frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{2}\ln(x+1) + C$$

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حاول أن تحل(5) أوجد :

$$\int x^2 \sin x \ dx$$

الحل :

$$\int udv = uv - \int vdu$$

$$u = x^2 \qquad dv = \sin x \, dx$$
$$du = 2xdx \qquad v = -cosx$$

$$\int x^2 \sin x \ dx = -x^2 \cos x + \int 2xcos\ x\, dx$$

$$u = 2x \qquad dv = \cos x \, dx$$
$$du = 2\ dx \qquad v = \sin x$$

$$\int 2xcos\ x\ dx = 2xsin\ x\ - \int 2 \sin x \, dx$$
$$= 2xsin\ x + 2cos\ x + C$$

$$\therefore \int x^2 \sin x \ dx = -x^2 \cos x + 2x\ sin\ x + 2 \cos x + C$$

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حاول أن تحل(6) أوجد :

$$\int x^2\ e^{x+2}\ dx$$

الحل :

$$\int udv = uv - \int vdu$$

$$u = x^2 \qquad dv = e^{x+2}dx$$
$$du = 2xdx \qquad v = e^{x+2}$$

$$\int x^2 e^{x+2}\ dx = x^2 e^{x+2} - \int 2xe^{x+2}\ dx$$

$$u = 2x \qquad dv = e^{x+2}dx$$
$$du = 2\ dx \qquad v = e^{x+2}$$

$$\int 2xe^{x+2}\ dx = 2xe^{x+2}\ - \int 2e^{x+2}dx$$
$$= 2xe^{x+2} - 2e^{x+2} + C$$

$$\therefore \int x^2 e^{x+2}\ dx = x^2 e^{x+2} - 2x\ e^{x+2} + 2\ e^{x+2} + C$$

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حاول أن تحل(7) أوجد :

$$\int e^x \cos x \; dx$$

الحل :

$$\int udv = uv - \int vdu$$

$u = e^x$, $dv = \cos x \, dx$

$du = e^x dx$, $v = \sin x$

$$\int e^x \cos x \; dx = e^x \sin x - \int e^x \sin x \, dx$$

$u = e^x$, $dv = \sin x \, dx$

$du = e^x \, dx$, $v = -\cos x$

$$\int e^x \sin x \; dx = -e^x \cos x + \int e^x \cos x \, dx$$

$$\therefore \int e^x \cos x \; dx = e^x \sin x - \int e^x \sin x \, dx$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\therefore 2\int e^x \cos x \; dx = e^x(\sin x + \cos x) + C$$

$$\therefore \int e^x \cos x \; dx = \frac{1}{2} e^x(\sin x + \cos x) + C$$

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انتهت حلول حاول أن تحل

البند 5-5 : التكامل بالتجزئ

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5-6 التكامل باستخدام الكسور الجزئية

حاول أن تحل (1) لتكن الدالة f : $f(x) = \frac{2x-1}{x^2-4x+3}$

أوجد : (a) **الكسور الجزئية** . (b) $\int f(x)dx$

الحل :

(a) $$\frac{2x-1}{x^2-4x+3} = \frac{2x-1}{(x-3)(x-1)} = \frac{A_1}{x-3} + \frac{A_2}{x-1}$$

نضرب طرفي المعادلة في $(x-3)(x-1)$ **و نبسط ثم نعوض عن** x **بـ** 3 **ثم عن** x **بـ** 1

$$2x-1 = A_1(x-1) + A_2(x-3)$$

$$2(3)-1 = A_1(3-1) \Rightarrow A_1 = \frac{5}{2}$$

$$2(1)-1 = A_2(1-3) \Rightarrow A_2 = \frac{-1}{2}$$

نعوض عن A_1, A_2 **بقيمتيهما :**

$$\frac{2x-1}{x^2-4x+3} = \frac{\frac{5}{2}}{x-3} + \frac{\frac{-1}{2}}{x-1}$$

(b) $$\int f(x)\,dx = \int \left(\frac{\frac{5}{2}}{x-3} - \frac{\frac{1}{2}}{x-1} \right) dx$$

$$= \int \frac{\frac{5}{2}}{x-3} dx - \int \frac{\frac{1}{2}}{x-1} dx = \frac{5}{2} \int \frac{1}{x-3} dx - \frac{1}{2} \int \frac{1}{x-1} dx$$

$$= \frac{5}{2} \ln|x-3| - \frac{1}{2} \ln|x-1| + C$$

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حاول أن تحل(2)

أوجد : $\displaystyle\int \frac{x^2-2}{2x^3-5x^2-3x}dx$

الحل :

$$2x^3-5x^2-3x = x(2x^2-5x-3) = x(2x+1)(x-3)$$

$$\frac{x^2-2}{2x^3-5x^2-3x} = \frac{x^2-2}{x(2x+1)(x-3)} = \frac{A_1}{x}+\frac{A_2}{2x+1}+\frac{A_3}{x-3}$$

نضرب طرفي المعادلة في $x(2x+1)(x-3)$ **و نبسطّ ثم نعوض عن** x **بـ** 0 **ثم عن** x **بـ** $\frac{-1}{2}$

ثم عن x **بـ** 3

$$x^2-2 = A_1(2x+1)(x-3)+A_2x(x-3)+A_3x(2x+1)$$

$$(0)^2-2 = A_1(2(0)+1)\big((0)-3\big) \Rightarrow A_1 = \frac{2}{3}$$

$$\left(\frac{-1}{2}\right)^2-2 = A_2\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-3\right) \Rightarrow A_2 = -1$$

$$3^2-2 = A_3(3)(2(3)+1) \Rightarrow A_3 = \frac{1}{3}$$

$$\therefore \frac{x^2-2}{2x^3-5x^2-3x} = \frac{\frac{2}{3}}{x}+\frac{-1}{2x+1}+\frac{\frac{1}{3}}{x-3}$$

b $$\int \frac{x^2-2}{2x^3-5x^2-3x}dx = \int \frac{\frac{2}{3}}{x}+\frac{-1}{2x+1}+\frac{\frac{1}{3}}{x-3}dx$$

$$= \frac{2}{3}\int \frac{1}{x}dx - \int \frac{1}{2x+1}dx + \frac{1}{3}\int \frac{1}{x-3}dx$$

$$= \frac{2}{3}\ln|x| - \frac{1}{2}\ln|2x+1| + \frac{1}{3}\ln|x-3| + C$$

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حاول أن تحل(3)

أوجد : $\int \frac{4x^2-4x+1}{x^3-2x^2+x}dx$

الحل :

$$x^3-2x^2+x = x(x^2-2x+1) = x(x-1)^2$$

$$\therefore \frac{4x^2-4x+1}{x^3-2x^2+x} = \frac{4x^2-4x+1}{x(x-1)^2} = \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2}$$

نضرب طرفي المعادلة في $x(x-1)^2$ **و نبسّط ثم نعوض عن** x **بـ** 0 **ثم عن** x **بـ** 1

$$4x^2-4x+1 = A_1(x-1)^2 + A_2x(x-1) + A_3x$$

$$4(0)^2-4(0)+1 = A_1(0-1)^2 + A_2(0)(0-1) + A_3(0) \Rightarrow A_1 = 1$$

$$4(1)^2-4(1)+1 = A_1(1-1)^2 + A_2(1)(1-1) + A_3(1) \Rightarrow A_3 = 1$$

نعوض بالمعادلة عن $A_1 = 1$ **و** $A_3 = 1$ **وعن** x **بـ** 2 **(مثلاً)**

$$4(2)^2-4(2)+1 = (1)(2-1)^2 + A_2(2)(2-1) + (1)(2) \Rightarrow A_2 = 3$$

$$\therefore \frac{4x^2-4x+1}{x^3-2x^2+x} = \frac{1}{x} + \frac{3}{x-1} + \frac{1}{(x-1)^2}$$

$$\int \frac{4x^2-4x+1}{x^3-2x^2+x}dx = \int \left(\frac{1}{x} + \frac{3}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \int \frac{1}{x}dx + \int \frac{3}{x-1}dx + \int \frac{1}{(x-1)^2}dx$$

$$= \ln|x| + 3\ln|x-1| - \frac{1}{x-1} + C$$

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حاول أن تحل(4)

أوجد : $\int \frac{x^2+1}{x^3+4x^2}dx$

الحل :

$$x^3+4x^2 = x^2(x+4)$$

$$\therefore \frac{x^2+1}{x^3+4x^2} = \frac{x^2+1}{x^2(x+4)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{(x+4)}$$

نضرب طرفي المعادلة في $x^2(x+4)$ **و نبسط ثمّ نعوض عن** x **بـ** 0 **ثم عن** x **بـ** -4

$$x^2+1 = A_1x(x+4) + A_2(x+4) + A_3x^2$$

$$(0)^2+1 = A_1(0)(0+4) + A_2(0+4) + A_3(0)^2 \Rightarrow A_2 = \frac{1}{4}$$

$$(-4)^2+1 = A_1(-4)(-4+4) + A_2(-4+4) + A_3(-4)^2 \Rightarrow A_3 = \frac{17}{16}$$

نعوض بالمعادلة عن $A_2 = \frac{1}{4}$ **و** $A_3 = \frac{17}{16}$ **وعن** x **بـ** 1 **(مثلاً)**

$$(1)^2+1 = A_1(1)(1+4) + \frac{1}{4}(1+4) + \frac{17}{16}(1)^2 \Rightarrow A_1 = \frac{-1}{16}$$

$$\therefore \frac{x^2+1}{x^3+4x^2} = \frac{\frac{-1}{16}}{x} + \frac{\frac{1}{4}}{x^2} + \frac{\frac{17}{16}}{(x+4)}$$

$$\int \frac{x^2+1}{x^3+4x^2}dx = \int \left(\frac{\frac{-1}{16}}{x} + \frac{\frac{1}{4}}{x^2} + \frac{\frac{17}{16}}{(x+4)} \right) dx$$

$$= \frac{-1}{16}\int \frac{1}{x}dx + \frac{1}{4}\int \frac{1}{x^2}dx + \frac{17}{16}\int \frac{1}{(x+4)}dx$$

$$= \frac{-1}{16}\ln|x| - \frac{1}{4x} + \frac{17}{16}\ln|x+4| + C$$

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حاول أن تحل(5) أوجد :

$$\int \frac{x^2-3x+7}{x^2-4x+4}\,dx$$

الحل :

ⓐ $$\int \frac{x^2-3x+7}{x^2-4x+4}\,dx = \int \frac{x^2-3x+7}{(x-2)^2}\,dx = \int (x^2-3x+7)(x-2)^{-2}\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$$u = x^2-3x+7 \qquad dv = (x-2)^{-2}dx$$

$$du = (2x-3)dx \qquad v = \frac{-1}{x-2}$$

$$\int \frac{x^2-3x+7}{x^2-4x+4}\,dx = (x^2-3x+7)\left(\frac{-1}{x-2}\right) - \int \left(\frac{-1}{x-2}\right)(2x-3)dx$$

$$= \left(\frac{-(x^2-3x+7)}{x-2}\right) + \int \left(\frac{2x-4}{x-2} + \frac{1}{x-2}\right)dx$$

$$= \left(\frac{-(x^2-3x+7)}{x-2}\right) + \int 2dx + \int \frac{1}{x-2}dx$$

$$= \left(\frac{-(x^2-3x+7)}{x-2}\right) + 2x + \ln|x-2| + C$$

طريقة ثانية

ⓐ $$\int \frac{x^2-3x+7}{x^2-4x+4}\,dx = \int \frac{x^2-4x+4+x+3}{x^2-4x+4}\,dx = \int \left(1 + \frac{x+3}{x^2-4x+4}\right)dx$$

$$= \int \left(1 + \frac{x-2+5}{x^2-4x+4}\right)dx = \int \left(1 + \frac{x-2}{x^2-4x+4} + \frac{5}{x^2-4x+4}\right)dx$$

$$= \int 1dx + \int \frac{1}{x-2}dx + 5\int \frac{1}{(x-2)^2}dx$$

$$= x + \ln|x-2| - \frac{5}{x-2} + C$$

(b) $\int \frac{x^3-2x^2-4}{x^3-2x^2}dx$

∴ درجة البسط = درجة المقام

∴ نقسم البسط على المقام باستخدام القسمة المطولة

$$\begin{array}{r|l} & 1 \\ x^3-2x^2 & x^3-2x^2+\quad -4 \\ - & x^3-2x^2 \\ \hline & \qquad\qquad -4 \end{array}$$

$$\frac{x^3-2x^2-4}{x^3-2x^2} = 1-\frac{4}{x^3-2x^2}$$

$$\frac{4}{x^3-2x^2} = \frac{4}{x^2(x-2)} = \frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x-2}$$

نضرب بـ $x^2(x-2)$

$$4 = A_1x(x-2)+A_2(x-2)+A_3(x^2)$$

نضع x **بـ** 0

$$4 = A_1(0)(0-2)+A_2(0-2)+A_3(0^2) \Longrightarrow A_2=-2$$

نضع x **بـ** 2

$$4 = A_1(2)(2-2)+A_2(2-2)+A_3(2^2) \Longrightarrow A_3=1$$

بالتعويض عن $A_2=-2$ ، $A_3=1$ **و إحدى قيم** x **ولتكن** 1

$$4 = A_1(1)(1-2)+(-2)(1-2)+(1)(1^2) \Longrightarrow A_1=-1$$

$$\int \frac{x^3-2x^2-4}{x^3-2x^2}dx = \int\left[1-\left(\frac{-1}{x}+\frac{-2}{x^2}+\frac{1}{x-2}\right)\right]dx$$

$$= \int 1dx + \int \frac{1}{x}dx + \int \frac{2}{x^2}dx - \int \frac{1}{x-2}dx$$

$$= x+\ln|x|-\frac{2}{x}-\ln|x-2|+C$$

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حاول أن تحل (6)

أوجد : $\int \frac{x^3-7x+9}{x^2-3x+2}dx$

∴ **درجة البسط > درجة المقام**

∴ **نقسم البسط على المقام باستخدام القسمة المطولة**

$$\begin{array}{r|l} & x+3 \\ x^2-3x+2 & x^3 + \quad -7x+9 \\ - & x^3-3x^2+2x \\ & 3x^2-9x+9 \\ - & 3x^2-9x+6 \\ & 3 \end{array}$$

$$\frac{x^3-7x+9}{x^2-3x+2} = (x+3) + \frac{3}{x^2-3x+2}$$

$$\frac{3}{x^2-3x+2} = \frac{3}{(x-2)(x-1)} = \frac{A_1}{x-2} + \frac{A_2}{x-1}$$

نضرب بـ $(x-2)(x-1)$

$$3 = A_1(x-1) + A_2(x-2)$$

نضع x **بـ** 2

$$3 = A_1(2-1) + A_2(2-2) \Longrightarrow A_1 = 3$$

نضع x **بـ** 1

$$3 = A_1(1-1) + A_2(1-2) \Longrightarrow A_2 = -3$$

$$\frac{3}{x^2-3x+2} = \frac{3}{x-2} + \frac{-3}{x-1}$$

$$\int \frac{x^3-7x+9}{x^2-3x+2}dx = \int \left[(x+3) - \left(\frac{3}{x-2} + \frac{-3}{x-1}\right)\right]dx$$

$$= \int (x+3)dx + 3\int \frac{1}{x-2}dx - 3\int \frac{1}{x-1}dx$$

$$= \frac{x^2}{2} + 3x + 3\ln|x-2| - 3\ln|x-1| + C$$

حاول أن تحل (7) : أوجد : $\int \frac{2x^4+3x^2-7}{x^3-6x^2+9x}dx$

∴ درجة البسط > درجة المقام ∴ نقسم البسط على المقام باستخدام القسمة المطولة

$$\begin{array}{r|l} & 2x+12 \\ \hline x^3-6x^2+9x & 2x^4 + \quad +3x^2 + \quad -7 \\ - & 2x^4-12x^3+18x^2 \\ \hline & 12x^3-15x^2+ \quad -7 \\ - & 12x^3-72x^2+180x \\ \hline & 57x^2-108x-7 \end{array}$$

$$\therefore \frac{2x^4+3x^2-7}{x^3-6x^2+9x} = (2x+12) + \frac{57x^2-108x-7}{x^3-6x^2+9x}$$

$$\frac{57x^2-108x-7}{x^3-6x^2+9x} = \frac{57x^2-108x-7}{x(x-3)^2} = \frac{A_1}{x} + \frac{A_2}{x-3} + \frac{A_3}{(x-3)^2}$$

نضرب بـ $x(x-3)^2$

$$57x^2-108x-7 = A_1(x-3)^2 + A_2x(x-3) + A_3x$$

نضع x **بـ** 0 $\qquad -7 = 9A_1 \Longrightarrow A_1 = \frac{-7}{9}$

نضع x **بـ** 3 $\qquad 182 = 3A_3 \Longrightarrow A_3 = \frac{182}{3}$

نعوض عن $A_1 = \frac{-7}{9}$ ، $A_3 = \frac{182}{3}$ ، $x = 1$ **نحصل :** $A_2 = \frac{520}{9}$

$$\frac{57x^2-108x-7}{x^3-6x^2+9x} = \frac{\frac{-7}{9}}{x} + \frac{\frac{520}{9}}{x-3} + \frac{\frac{182}{3}}{(x-3)^2}$$

$$\int \frac{2x^4+3x^2-7}{x^3-6x^2+9x}dx = \int \left[(2x+12) - \left(\frac{\frac{-7}{9}}{x} + \frac{\frac{520}{9}}{x-3} + \frac{\frac{182}{3}}{(x-3)^2}\right)\right]dx$$

$$= \int (2x+12)dx - \frac{7}{9}\int \frac{1}{x}dx + \frac{520}{9}\int \frac{1}{x-3}dx + \frac{182}{3}\int \frac{1}{(x-3)^2}dx$$

$$= x^2 + 12x - \frac{7}{9}\ln|x| + \frac{520}{9}\ln|x-3| - \frac{182}{3(x-3)} + C$$

******انتهت حلول حاول أن تحل (التكامل باستخدام الكسور الجزئية) ******

5-7 التكامل المحدد

حاول أن تحل(1) : أوجد :

$$\int_{2}^{7}(x^3-2x^2+2)dx$$

الحل :

$$\int_{2}^{7}(x^3-2x^2+2)dx=\left[\frac{1}{4}x^4-\frac{2}{3}x^3+2x\right]_2^7$$

$$=\left(\frac{1}{4}(7)^4-\frac{2}{3}(7)^3+2(7)\right)-\left(\frac{1}{4}(2)^4-\frac{2}{3}(2)^3+2(2)\right)=\frac{4595}{12}$$

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حاول أن تحل(2) أوجد :

a $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{1}{2}\sin 2x\ -csc^2x\right)dx$$

الحل :

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{1}{2}\sin 2x\ -csc^2x\right)dx=\left[-\frac{1}{4}\cos 2x-(-\cot x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=\left(-\frac{1}{4}\cos 2\left(\frac{\pi}{2}\right)-(-\cot\frac{\pi}{2})\right)-\left(-\frac{1}{4}\cos 2\left(\frac{\pi}{4}\right)-(-\cot\frac{\pi}{4})\right)$$

$$=\left(-\frac{1}{4}(-1)+0\right)-\left(-\frac{1}{4}(0)+1\right)=-\frac{3}{4}$$

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b $$\int_{2}^{-3}5dx=[5x]_2^{-3}=5(-3-2)=-25$$

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c $$\int_{3}^{3}(-2x^3+x^2)dx=0$$

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(d) $\int_{2}^{4} \frac{dx}{x-1} = [\ln|x-1|\,]_{2}^{4} = \ln|4-1| - \ln|2-1| = \ln 3 - 0 = \ln 3$

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حاول أن تحل(3) أوجد :

(a) $\int_{-3}^{4} |2x-4|dx = \int_{-3}^{2} |2x-4|dx + \int_{2}^{4} |2x-4|dx$

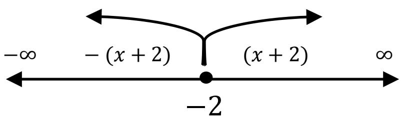
$= \int_{-3}^{2} -(2x-4)dx + \int_{2}^{4} (2x-4)dx$

$= \int_{-3}^{2} (4-2x)dx + \int_{2}^{4} (2x-4)dx$

$= [4x-x^2]_{-3}^{2} + [x^2-4x]_{2}^{4}$

$= [(8-4)-(-12-9)] + [(16-16)-(4-8)] = 29$

(b) $\int_{1}^{3} |x+2|dx = \int_{1}^{3} (x+2)dx = \left[\frac{1}{2}x^2 + 2x\right]_{1}^{3}$



$= \left(\frac{1}{2}(3)^2 + 2(3)\right) - \left(\frac{1}{2}(1)^2 + 2(1)\right) = 8$

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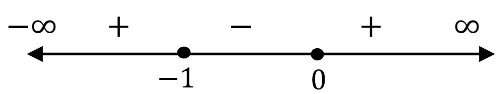
حاول أن تحل(4) دون حساب قيمة التكامل أثبت أن :

$\int_{-1}^{0} (x^2+x)dx \le 0$

الحل :

$f(x) = x^2 + x$

$x^2 + x = 0 \Rightarrow x(x+1) = 0 \Rightarrow x = 0\,, x = -1$



$\therefore f(x) \le 0 \quad , \ \forall\, x \in [-1,0]$

$\therefore x^2 + x \le 0 \quad , \ \forall\, x \in [-1,0]$ $\qquad \therefore \int_{-1}^{0} (x^2+x)dx \le 0$

حاول أن تحل(5) دون حساب قيمة التكامل أثبت أن :

$$\int_{-1}^{2}(x^2+1)dx \geq \int_{-1}^{2}(x-1)dx$$

الحل :

نفرض أن $g(x) = x - 1 \quad , \quad f(x) = x^2 + 1$

$$f(x) - g(x) = (x^2+1) - (x-1) = x^2 + 1 - x + 1 = x^2 - x + 2$$

نضع $x^2 - x + 2 = 0$

$$\Delta = b^2 - 4ac = (-1)^2 - 4(1)(2) = -7 < 0$$

$\therefore$ **لاتوجد جذور حقيقية للمعادلة** $\Leftarrow$ $f(x) - g(x)$ **وحيدة الإشارة وبأخذ قيمة اختيارية**

$$f(x) - g(x) \geq 0 \quad , \ \forall\, x \in \mathbb{R}$$

$$\therefore f(x) - g(x) \geq 0 \quad , \ \forall\, x \in [-1,2]$$

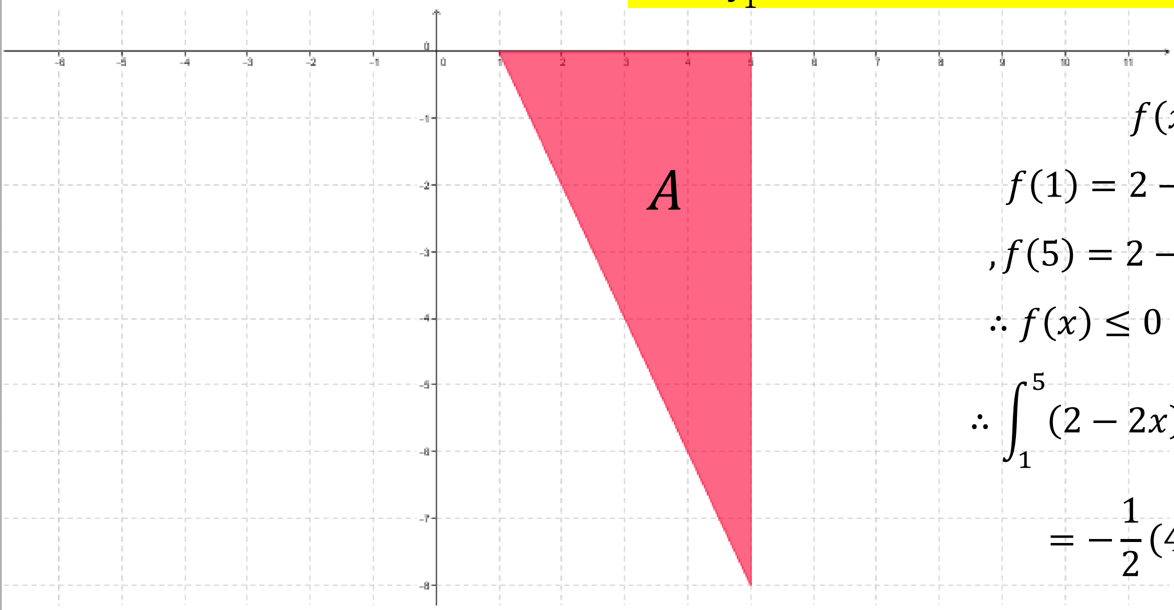
$$\therefore (x^2+1) - (x-1) \geq 0 \quad , \ \forall\, x \in [-1,2] \implies (x^2+1) \geq (x-1)$$

$$\therefore \int_{-1}^{2}(x^2+1)dx \geq \int_{-1}^{2}(x-1)dx$$

****

حاول أن تحل(6) أوجد قيمة : $\int_{1}^{5}(2-2x)dx$ **بيانيا**

الحل :



$$f(x) = 2 - 2x$$

$$f(1) = 2 - 2(1) = 0$$

$$, f(5) = 2 - 2(5) = -8$$

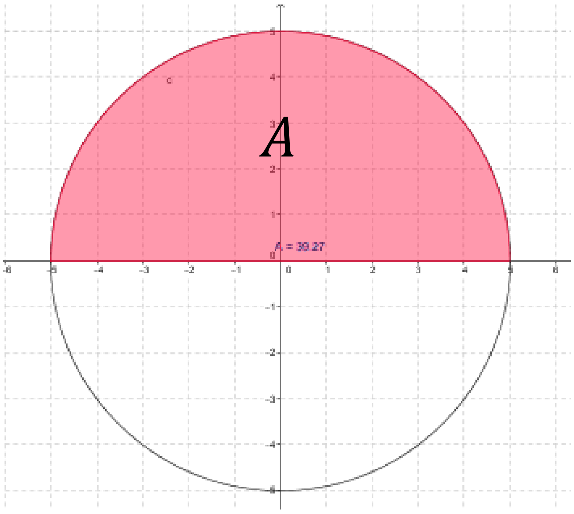
$$\therefore f(x) \leq 0 \ \forall\, x \in [1,5]$$

$$\therefore \int_{1}^{5}(2-2x)dx = -A$$

$$= -\frac{1}{2}(4)(8) = -16$$

حاول أن تحل(7) أوجد :

a $\int_{-5}^{5} \sqrt{25 - x^2}\, dx$



نأخذ $y = \sqrt{25 - x^2}$

$\therefore y^2 = 25 - x^2$

$\therefore y^2 + x^2 = 25$

وهي معادلة دائرة مركزها نقطة الأصل

و طول نصف قطرها 5 وحدة طول

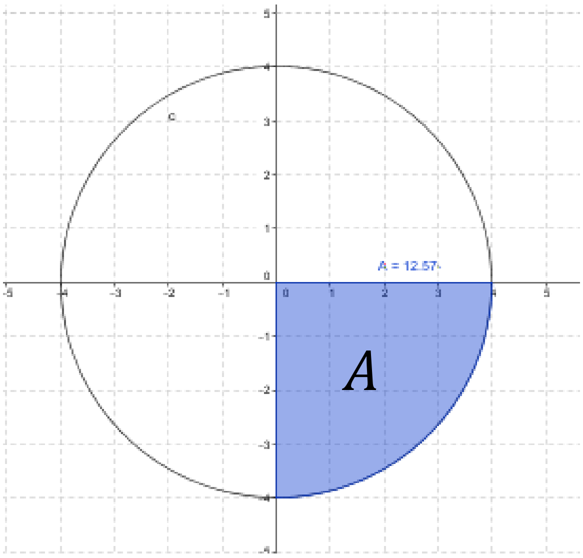
والدالة : $y = \sqrt{25 - x^2}$

تمثل معادلة النصف العلوي للدائرة

$$\therefore \int_{-5}^{5} \sqrt{25 - x^2}\, dx = A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(5)^2 = \frac{25}{2}\pi$$

---

b $\int_{0}^{4} -\sqrt{16 - x^2}\, dx$



نأخذ $y = -\sqrt{16 - x^2}$

$\therefore y^2 = 16 - x^2$

$\therefore y^2 + x^2 = 16$

وهي معادلة دائرة مركزها نقطة الأصل

و طول نصف قطرها 4 وحدة طول

والدالة : $y = -\sqrt{16 - x^2}$

تمثل معادلة النصف السفلي للدائرة

$$\therefore \int_{0}^{4} -\sqrt{16 - x^2}\, dx = -A = -\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(4)^2 = -4\pi$$

****

حاول ان تحل (8) ⓐ **هل يمكن حل مثال(8) بطريقة أخرى ؟ فسّر إجابتك .**

الحل :

$$\int_0^{\frac{\pi}{4}} \tan x \ sec^2 x \, dx$$

$$f(x) = \tan x \Longrightarrow f'(x) = sec^2 x$$

$$\int_0^{\frac{\pi}{4}} f(x) \ f'(x) dx = \left[\frac{1}{2}\left(f(x)\right)^2\right]_0^{\frac{\pi}{4}} = \left[\frac{1}{2}(\tan x)^2\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\left(\tan\frac{\pi}{4}\right)^2 - (\tan 0)^2\right] = \frac{1}{2}(1-0) = \frac{1}{2}$$

ⓑ **أوجد :** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \cos 2x \, dx$

الحل :

$$u = \sin 2x \Longrightarrow du = 2\cos 2x \, dx \Longrightarrow \cos 2x \, dx = \frac{1}{2} du$$

عندما $x = \frac{\pi}{6}$ **فإن** $u = \frac{\sqrt{3}}{2}$ ، **عندما** $x = \frac{\pi}{3}$ **فإن** $u = \frac{\sqrt{3}}{2}$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \cos 2x \, dx = \int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} u . \frac{1}{2} du = 0$$

طريقة أخرى : $f(x) = \sin 2x \Longrightarrow f'(x) = 2\cos 2x$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \cos 2x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f(x) . \frac{1}{2} f'(x) \, dx = \frac{1}{4}\left[\left(f(x)\right)^2\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{4}[(\sin 2x)^2]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{4}\left[\left(\sin\frac{2\pi}{3}\right)^2 - \left(\sin\frac{\pi}{3}\right)^2\right] = 0$$

طريقة أخرى : $\sin 2x . cos 2x = \frac{1}{2}\sin 4x$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \cos 2x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} \sin 4x \, dx = \left[-\frac{1}{8}\cos 4x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 0$$

حاول ان تحل (9)

a) $\int_{-1}^{1}\left((x+1)\sqrt{x^2+2x+5}\right)dx$

الحل :

$$u = x^2+2x+5 \Rightarrow du = (2x+2)dx \Rightarrow (x+1)dx = \frac{1}{2}du$$

عندما $x=-1$ فإن $u=4$ ، عندما $x=1$ فإن $u=8$

$$\int_{-1}^{1}\left((x+1)\sqrt{x^2+2x+5}\right)dx = \frac{1}{2}\int_{4}^{8} u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\times\frac{2}{3}\left[u^{\frac{3}{2}}\right]_4^8 = \frac{1}{3}\left[8^{\frac{3}{2}} - 4^{\frac{3}{2}}\right] = 4.8758$$

---

b) $\int_{2}^{5} x\sqrt{x-1}dx$

الحل :

$$\int_{2}^{5} x\sqrt{x-1}\,dx = \int_{2}^{5}\left((x-1)+1\right)(x-1)^{\frac{1}{2}}dx$$

$$= \int_{2}^{5}(x-1)^{\frac{3}{2}}\,dx + \int_{2}^{5}(x-1)^{\frac{1}{2}}\,dx$$

$$= \frac{2}{5}\left[(x-1)^{\frac{5}{2}}\right]_2^5 + \frac{2}{3}\left[(x-1)^{\frac{3}{2}}\right]_2^5$$

$$= \frac{2}{5}\left[(4)^{\frac{5}{2}} - (1)^{\frac{5}{2}}\right] + \frac{2}{3}\left[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] = \frac{256}{15}$$

--- طريقة ثانية ---- باستخدام التكامل بالتجزئ ----

وذلك بوضع : $dv = \sqrt{x-1} = (x-1)^{\frac{1}{2}}$ $\qquad u = x$

$v = \frac{2}{3}(x-1)^{\frac{3}{2}}$ $\qquad du = dx$

ثم نتابع على القاعدة :

$$\int_{2}^{5} u\,dv = u.v - \int_{2}^{5} v\,du$$

حاول أن تحل (10)

أوجد : $\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$

الحل :

$u = x \qquad dv = sec^2 x\,dx$

$du = dx \qquad v = \tan x$

$$\int_0^{\frac{\pi}{4}} u\,dv = u.v - \int_0^{\frac{\pi}{4}} v\,du$$

$$\int_0^{\frac{\pi}{4}} x sec^2 x\,dx = [x\tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,dx$$

$$= \left[\frac{\pi}{4}\tan\frac{\pi}{4} - 0\right] - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx = \frac{\pi}{4} + \int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x}\,dx$$

$$= \frac{\pi}{4} + [\ln|\cos x|]_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \left(\ln\left|\cos\frac{\pi}{4}\right| - \ln\left|\cos\frac{\pi}{4}\right|\right) = \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}$$

****

حاول أن تحل (11)

أوجد : $\int_4^7 \frac{3x^2-17}{x^2-x-6}\,dx$

الحل :

القسمة المطولة:

$$\begin{array}{r|l} & 3 \\ \hline x^2-x-6 & 3x^2 \qquad - 17 \\ - & 3x^2 - 3x - 18 \\ \hline & 3x + 1 \end{array}$$

$$\frac{3x^2-17}{x^2-x-6} = 3 + \frac{3x+1}{x^2-x-6}$$

$$\frac{3x+1}{x^2-x-6} = \frac{3x+1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$3x+1 = A(x+2) + B(x-3)$$

$$x = 3 \Rightarrow A = 2 \quad , \quad x = -2 \Rightarrow B = 1$$

$$\frac{3x+1}{x^2-x-6} = \frac{2}{x-3} + \frac{1}{x+2}$$

$$\int_4^7 \frac{3x^2-17}{x^2-x-6}\,dx = \int_4^7 3\,dx + \int_4^7 \frac{2}{x-3}\,dx + \int_4^7 \frac{1}{x+2}\,dx$$

$$= [3x]_4^7 + 2[\ln|x-3|]_4^7 + [\ln|x+2|]_4^7$$

$$= 3(7-4) + 2[\ln 4 - \ln 1] + [\ln 9 - \ln 6] = 12.178$$

***************** **انتهت حلول التكامل المحدد** *****************

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انتهت حلول حاول أن تحل

البند 5-5: التكامل بالتجزئ